

The proof of the Dijkgraaf-Vafa conjecture and application to the mass gap and confinement problems

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ABSTRACT: Using generalized Konishi anomaly equations, it is known that one can express, in a large class of supersymmetric gauge theories, all the chiral operators expectation values in terms of a finite number of a priori arbitrary constants. We show that these constants are fully determined by the requirement of gauge invariance and an additional anomaly equation. The constraints so obtained turn out to be equivalent to the extremization of the Dijkgraaf-Vafa quantum glueball superpotential, with all terms (including the Veneziano-Yankielowicz part) unambiguously fixed. As an application, we fill non-trivial gaps in existing derivations of the mass gap and confinement properties in super Yang-Mills theories.

KEYWORDS: Nonperturbative Effects, Confinement, Gauge Symmetry, Supersymmetric gauge theory.

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1. General presentation

The aim of this paper is to present a general proof of the Dijkgraaf-Vafa quantum equations of motion [1],

$$\frac{\partial W_{\text{DV}}}{\partial s_I} = 0, \tag{1.1}$$

that determine the glueball superfields expectation values $s_I = \langle S_I \rangle$ in a large class of $\mathcal{N} = 1$ supersymmetric gauge theories. The holomorphic function $W_{\text{DV}}(s_I)$ is the sum of a suitable Veneziano-Yankielowicz term [2] and of an infinite power series in the s_I s. The equations (1.1) are generally interpreted in a low energy effective action framework: the gauge invariant composite fields S_I are assumed to be good low energy degrees of freedom, with low energy effective superpotential given by W_{DV} . The conditions (1.1) are then the dynamical equations of motion in the low energy effective theory. This interpretation is extremely difficult to justify from first principles in the gauge theory, as stressed for example in [3]. Our main result is to show that (1.1) actually follows from general consistency conditions in the gauge theory, in some sense analogous to the Dirac quantization condition. More precisely, we shall demonstrate that the vanishing of the differences $(\partial_{s_I} - \partial_{s_{\bar{I}}})W_{\text{DV}}$ follows from gauge invariance, and that the vanishing of the sum $\sum_I s_I \partial_{s_I} W_{\text{DV}}$ follows from a Ward identity. We thus interpret the equations (1.1) as being *non-dynamical identities*.

In the dual closed string theory formulation [4], W_{DV} is a genuine effective superpotential associated with the non-trivial background fluxes [5]. From this point of view, deriving (1.1) amounts to proving that the closed string background proposed in [4], and their generalizations based on the geometric transition picture (see [6] for non-trivial examples), are indeed the correct duals of the gauge theory. We thus have a dual interpretation of the equations (1.1): they are non-dynamical from the gauge theory, or open string point of view, and dynamical from the closed string point of view. This is a nice new example of a familiar phenomenon for dual formulations of the same theory (the basic example being the electric-magnetic duality of the Maxwell's equations, under which the Bianchi identities and the dynamical equations of motion are exchanged).

A startling feature of the equations (1.1) is that they provide a unifying framework to derive essentially all the known exact results in supersymmetric gauge theories. For example, they have been used to obtain the Seiberg-Witten solutions of $\mathcal{N} = 2$ theories [7], to study Seiberg dualities [8], or to find new results on the quantum space of parameters in $\mathcal{N} = 1$ theories [9–12]. In some cases, the same results can also be obtained by using various techniques (S-duality, singularity theory, integrable systems, geometric engineering, mirror symmetry, brane constructions, geometric transitions etc...), but at the expense of making several non-trivial assumptions. Actually, there are very few derivations from first principles. The one exception is Nekrasov's explicit summing-up of instanton series [13], which yields the Seiberg-Witten prepotentials in $\mathcal{N} = 2$ super Yang-Mills theories. The main application of the proof of (1.1) is to provide automatically assumption-free derivations from first principles of all the above mentioned exact results. For example, the derivation of the Seiberg-Witten $\mathcal{N} = 2$ prepotentials along these lines do not even assume that only instantons contribute, as in Nekrasov's, but *prove* it (see section 5). Another

important application is to the mass gap and confinement problems in $\mathcal{N} = 1$ super Yang-Mills. The main argument in favor of confinement was given by Seiberg and Witten in their classic papers [14]. The idea is to perturb a $\mathcal{N} = 2$ theory by adding a tree level superpotential. The argument relies on two main hypotheses: the solution of the $\mathcal{N} = 2$ theory is correct; the $\mathcal{N} = 1$ theory obtained by perturbing $\mathcal{N} = 2$ creates a mass gap. We are able to waive these hypotheses in section 5.

The paper is organized as follows. In section 2, we review some useful concepts and results, including the work of Cachazo, Douglas, Seiberg and Witten [3] on generalized Konishi anomaly equations. These equations are used in section 3 to study, for any adjoint field X in the theory, the quantum characteristic function

$$\mathcal{F}(z) = \langle \det(z - X) \rangle. \tag{1.2}$$

Our strategy is to show that gauge invariance puts general constraints on \mathcal{F} that turn out to be equivalent to $(\partial_{S_I} - \partial_{S_J})W_{\text{DV}} = 0$. In section 4, we find a Ward identity that provides the missing equation in the form $\sum_I S_I \partial_{S_I} W_{\text{DV}} = 0$. We also comment on the possible existence of Kovner-Shifman vacua [15]. Finally in section 5 we apply the results to the mass gap and confinement problems, giving en passant the derivation of the Seiberg-Witten solution for $\mathcal{N} = 2$. Our arguments are completely general, but for concreteness we mainly focus on the $\mathcal{N} = 1$ theory with gauge group $U(N)$, one adjoint matter superfield X and $N_f \leq 2N$ fundamental flavors. We also give details on the case with two adjoint fields, because it illustrates nicely some non-trivial aspects of the proof.

2. Introductory material and discussion

2.1 The quantum effective superpotential

A most important object in four dimensional $\mathcal{N} = 1$ supersymmetric gauge theories is the quantum effective superpotential W_{low} , which determines the F-term part of the quantum effective action. It is defined by a path integral over the vector and ghosts supermultiplets V and the matter chiral multiplets X , with boundary conditions determined by the choice of a supersymmetric vacuum $|0\rangle$,

$$e^{i \int d^4x (2N \text{Re} \int d^2\theta W_{\text{low}}^{(0)}(T_K) + D\text{-terms})} = \int_{|0\rangle} [dV dX] e^{i \int d^4x \mathcal{L}(V, X; T_K)}. \tag{2.1}$$

The normalization factor N is chosen for convenience to coincide with the number of colors of the gauge group $U(N)$. The T_K are external chiral superfields, with lowest components t_K , that couple to gauge invariant operators \mathcal{O}_K in such a way that

$$\langle 0 | \mathcal{O}_K | 0 \rangle = \frac{\partial W_{\text{low}}^{(0)}}{\partial t_K}. \tag{2.2}$$

In general, the knowledge of W_{low} is equivalent to the knowledge of the full set of chiral operator vacuum expectation values in the theory.

For example, consider the $U(N)$ gauge theory with one adjoint X , $N_f \leq 2N$ flavors of fundamentals Q_f and anti-fundamentals \tilde{Q}^f chiral multiplets, and tree level lagrangian

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left[\tau_{\text{UV}} \int d^2\theta \text{Tr}_F W^\alpha W_\alpha \right] + 2N \text{Re} \int d^2\theta \left(\text{Tr}_F W_{\text{tree}}(X) + {}^T \tilde{Q}^f m_f^{f'}(X) Q_{f'} \right) + \text{D-terms}. \quad (2.3)$$

The traces Tr_F are in the fundamental representation. The bare gauge coupling τ_{UV} at scale Λ_{UV} is given in terms of the dynamically generated scale Λ by

$$2i\pi\tau_{\text{UV}} = \ln \left(\frac{\Lambda}{\Lambda_{\text{UV}}} \right)^{2N-N_f}. \quad (2.4)$$

The one exception is when $N_f = 2N$, in which case $\tau_{\text{UV}} = \tau$ is not renormalized. The derivative of the tree-level superpotential is taken to be of the form

$$W'_{\text{tree}}(X) = \sum_{k=0}^d g_k X^k = g_d \prod_{I=1}^d (X - a_I). \quad (2.5)$$

The polynomials

$$m_f^{f'}(X) = \sum_{q \geq 0} m_{f,q}^{f'} X^q \quad (2.6)$$

are such that

$$\det_{f,f'} m(X) = U_0 \prod_{Q=1}^{\ell} (X - b_Q). \quad (2.7)$$

The most general vacuum

$$|N_1, k_1; \dots; N_d, k_d; \nu_1; \dots; \nu_\ell\rangle \quad (2.8)$$

is specified by the numbers N_I and ν_Q , $\nu_Q = 0$ or 1 , of eigenvalues of the matrix X that classically are equal to a_I and b_Q , with pattern of gauge symmetry breaking $U(N) \rightarrow U(N_1) \times \dots \times U(N_d)$, and by the integers k_I , defined modulo \mathbb{Z}_{N_I} , that correspond to low energy chiral symmetry breaking. In particular,

$$\sum_{I=1}^d N_I + \sum_{Q=1}^{\ell} \nu_Q = N. \quad (2.9)$$

In any vacuum $|N_I, k_I; \nu_Q\rangle$, the relations (2.2) read

$$\frac{1}{k+1} \langle \text{Tr}_F X^{k+1} \rangle = \frac{\partial W_{\text{low}}}{\partial g_k}, \quad \langle {}^T \tilde{Q}^f X^q Q_{f'} \rangle = \frac{\partial W_{\text{low}}}{\partial m_{f,q}^{f'}}, \quad (2.10)$$

$$\langle S \rangle = \frac{\partial W_{\text{low}}}{\partial \ln \Lambda^{2N-N_f}}, \quad (2.11)$$

where we have introduced the glueball superfield

$$S = -\frac{\text{Tr}_F W^\alpha W_\alpha}{16\pi^2 N}. \quad (2.12)$$

When $N_f = 2N$, $\ln \Lambda^{2N-N_f}$ is replaced by $2i\pi\tau$ in (2.11).

By a standard non-renormalization theorem, W_{low} is not corrected in perturbation theory, but there are in many cases non-trivial non-perturbative corrections, typically given by infinite sums over (fractional) instantons. The calculation of W_{low} is thus a difficult strong coupling problem.

2.2 The Dijkgraaf-Vafa approach

Dijkgraaf and Vafa made the remarkable conjecture [1] that the calculation of W_{low} in a large class of $\mathcal{N} = 1$ supersymmetric gauge theories could be reduced to a simple recipe. The recipe goes in two separate steps.

2.2.1 Step one: the matrix model

First one introduces a zero-dimensional gauged matrix model whose potential coincides with the tree-level superpotential of the gauge theory. For example, for the model (2.3), we use the one-matrix model coupled to flavors, for which the partition function \mathcal{F} is an integral over a $n \times n$ hermitian matrix X and n -vectors \tilde{Q}^f and $Q_{f'}$,

$$e^{n^2 \mathcal{F}/s^2} = \mathcal{N} \int dX dQ d\tilde{Q} e^{-\frac{n}{s} \text{Tr}_F W_{\text{tree}}(X) - \frac{n}{s} \tilde{Q}^f m_{f'}^{f'}(X) Q_{f'}}. \quad (2.13)$$

The normalization factor \mathcal{N} is independent on the couplings

$$t_K = (g_k, m_{f,q}^{f'}) \quad (2.14)$$

in the tree-level action and will be discussed later. The size n of the matrix and the vectors is a dummy variable and is not related to the number of colors N in the gauge theory. We are actually interested in the $n \rightarrow \infty$ limit, or more precisely in the sum over sphere and disk diagrams around a particular classical ($s \rightarrow 0$) solution, which yields an expansion of the form

$$\mathcal{F} = \mathcal{F}_s + \frac{s}{n} \mathcal{F}_d + \mathcal{O}(1/n^2). \quad (2.15)$$

The classical solutions are classified by the integers n_I and ν_Q , $\nu_Q = 0$ or 1 , where n_I and ν_Q give the number of eigenvalues of X equal to a_I and b_Q in the $s \rightarrow 0$ limit. In the $n \rightarrow \infty$ limit, the partition function, obtained by expanding around a particular matrix model vacuum

$$|n_1, \dots, n_d; \nu_1, \dots, \nu_\ell\rangle, \quad (2.16)$$

is a natural function of the filling fractions

$$s_I = s \frac{n_I}{n} \quad (2.17)$$

and of the ν_Q . Actually, since the sphere partition function \mathcal{F}_s is insensitive to the coupling to the fundamental flavors, it depends on the s_I only, whereas the disk partition function \mathcal{F}_d depends on both the s_I and the ν_Q .

We then introduce the Dijkgraaf-Vafa superpotential [1, 16]

$$W_{\text{DV}}(s_I; t_K) = - \sum_I N_I \frac{\partial \mathcal{F}_s}{\partial s_I} - \mathcal{F}_d. \quad (2.18)$$

The “weak” Dijkgraaf-Vafa conjecture states that

$$\frac{\partial W_{\text{DV}}}{\partial g_k} = \frac{1}{k+1} \langle \text{Tr}_F X^{k+1} \rangle, \quad \frac{\partial W_{\text{DV}}}{\partial m_{f,k}} = \langle {}^T \tilde{Q}^f X^k Q_{f'} \rangle, \quad (2.19)$$

where the ν_Q parametrizing the matrix model (2.16) and gauge theory (2.8) vacua are taken to be the same. The equations (2.19) determine an infinite set of chiral operators vacuum expectation values in terms of a finite number of undetermined constants s_I . These matrix model filling fractions are identified in the gauge theory with the gluino condensates in the $U(N_I)$ factors of the low energy gauge group,

$$s_I = \langle S_I \rangle, \quad (2.20)$$

or are taken to be zero if $N_I = 0$ (as explained in [3], the operators S_I are defined as in (2.12) with the insertion of a suitable projection operator).

2.2.2 Step two: the extremization of W_{DV}

The “strong” Dijkgraaf-Vafa conjecture states that

$$W_{\text{low}} = W_{\text{DV}}(s_I^0), \quad (2.21)$$

where the s_I^0 are the particular values of the s_I that solve the quantum equations of motion (1.1).

The function W_{DV} splits into a part W_{VY} that contains all the t_K -independent terms in W_{DV} and the t_K -dependent part that we denote W_{MM} ,

$$W_{\text{DV}}(s_I; t_K) = W_{\text{VY}}(s_I) + W_{\text{MM}}(s_I; t_K), \quad (2.22)$$

It is important to realize that the weak Dijkgraaf-Vafa conjecture does not depend on the form of W_{VY} , since any t_K -independent term drops from (2.19). On the other hand, W_{VY} plays a crucial rôle in the equations (1.1). A central result of the present work will be to derive, in sections 3 and 4, the form of W_{VY} from first principles.

With our definition (2.13), W_{VY} depends on the normalization factor \mathcal{N} . A priori, this normalization factor may be a complicated function of s and of the dynamically generated scale Λ . For example, for the model (2.3), we may postulate $\mathcal{N} \propto \Lambda^a s^b$ with $a + 3b = -n^2 - 2nN_f$ for dimensional reasons. The correct choice (that we will be able to justify later) turns out to be

$$\mathcal{N} \propto \Lambda^{-n^2 + nN_f} s^{-nN_f}. \quad (2.23)$$

There can be an additional purely numerical factor (independent of the t_K , s_I or Λ), that can be used to set a convenient normalization of Λ .

A last important comment is that with the choice (2.23), (2.13) and (2.18) would imply that $W_{\text{VY}} = -\sum_I N_I s_I \ln s_I$, which is slightly misleading. The correct formula is

$$W_{\text{VY}}(s_I) = -\sum_I s_I \ln s_I^{N_I}. \quad (2.24)$$

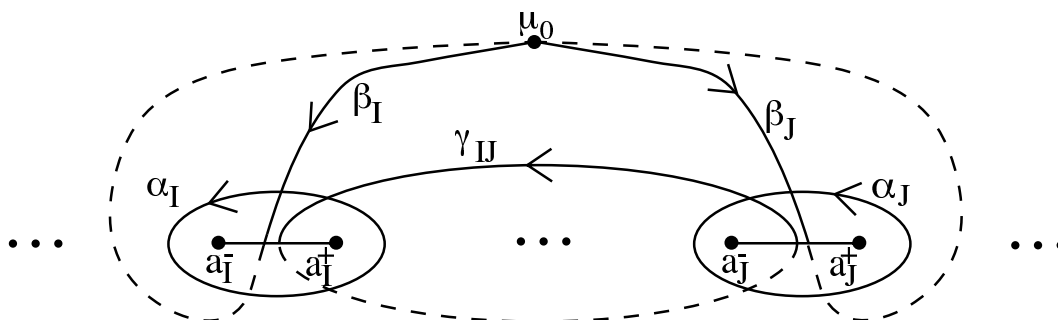


Figure 1: The non-compact two-sheeted Riemann surface \mathcal{C} , with the contours α_I , β_I and γ_{IJ} used in the main text. The contours β_I are defined modulo an integer multiple of α_I . The points $z = \mu_0$ on the first and second sheets are taken to infinity.

With this prescription, solving (1.1) automatically yields the full set of chirally asymmetric vacua (2.8) labeled by the integers k_I . Let us note that W_{VY} is the sum of the independent pure gauge theory contributions postulated in [2] for each factor $U(N_I)$ of the low energy gauge group. For this reason, it is often called the Veneziano-Yankielowicz part of the Dijkgraaf-Vafa superpotential.

2.3 The generalized Konishi anomaly equations

The weak form of the Dijkgraaf-Vafa conjecture was explained in a seminal paper by Cachazo, Douglas, Seiberg and Witten [3, 17].¹ These authors showed that the loop equations of the matrix model, that determine all the matrix model correlators as a function of the filling fractions s_I , were in one-to-one correspondence with a set of generalized Konishi anomaly equations in the gauge theory, the precise form of the correspondence being equivalent to (2.19).

The consequences of the generalized Konishi anomaly equations, and in particular of the equations (2.19), for the model (2.3) can be summarized as follows. Let us introduce the generating functions for the $\langle \text{Tr}_F X^k \rangle$, $\langle \text{Tr}_F W^\alpha W_\alpha X^k \rangle$ and $\langle {}^T \tilde{Q}^{f'} X^q Q_f \rangle$,

$$\mathcal{R}(z) = \left\langle \text{Tr}_F \frac{1}{z - X} \right\rangle, \tag{2.25}$$

$$\mathcal{S}(z) = -\frac{1}{16\pi^2 N} \left\langle \text{Tr}_F \frac{W^\alpha W_\alpha}{z - X} \right\rangle, \tag{2.26}$$

$$\mathcal{G}_f^{f'}(z) = \left\langle {}^T \tilde{Q}^{f'} \frac{1}{z - X} Q_f \right\rangle, \tag{2.27}$$

and the convenient combination

$$\mathcal{Y}(z) = W'_{\text{tree}}(z) - 2\mathcal{S}(z). \tag{2.28}$$

The function \mathcal{Y} satisfies a quadratic equation

$$\mathcal{Y}(z)^2 = W'_{\text{tree}}(z)^2 - \Delta(z), \tag{2.29}$$

¹Unfortunately, a proper non-perturbative discussion of the arguments in [3] has not yet appeared in the literature. In the present paper we focus on the strong form of the Dijkgraaf-Vafa conjecture, assuming that the weak form of the conjecture, or equivalently the results of [3], are correct.

where Δ is a polynomial of degree $\deg \Delta \leq d - 1 = \deg W'_{\text{tree}} - 1$. In a vacuum with low energy gauge group $U(N_1) \times \dots \times U(N_{\tilde{d}})$, $1 \leq \tilde{d} \leq d$, the following factorization in terms of a degree $d - \tilde{d}$ polynomial $H_{d-\tilde{d}}$ takes place,

$$W'^2_{\text{tree}}(z) - \Delta(z) = H^2_{d-\tilde{d}}(z) \prod_{I=1}^{\tilde{d}} (z - a_I^-)(z - a_I^+), \quad (2.30)$$

and thus \mathcal{Y} and \mathcal{S} are meromorphic functions on the genus $\tilde{d} - 1$ hyperelliptic surface

$$\mathcal{C} : y^2 = \prod_{I=1}^{\tilde{d}} (z - a_I^-)(z - a_I^+). \quad (2.31)$$

This two-sheeted Riemann surface, with some contours defined on it, is depicted in figure 1. The polynomial Δ , and thus the surface \mathcal{C} , is completely determined by the factorization condition (2.30) up to \tilde{d} parameters that can be conveniently chosen to be the α_I -period integrals of $\mathcal{S} dz$,

$$\frac{1}{2i\pi} \oint_{\alpha_I} \mathcal{S} dz = s_I. \quad (2.32)$$

In particular, we have

$$\sum_I s_I = s = \langle S \rangle, \quad (2.33)$$

the expectation value of the glueball superfield defined in (2.12). In the classical limit, $s_I \rightarrow 0$ for all I and we have $a_I^- \simeq a_I^+ \simeq a_I$. Equations (2.28) and (2.29) also show that $\mathcal{S}(z)$ is the sphere matrix model resolvent,

$$\mathcal{S}(z) = \left\langle \left\langle \frac{s}{n} \text{Tr}_F \frac{1}{z - X} \right\rangle \right\rangle_{\text{sphere}}, \quad (2.34)$$

where the double bracket notation is used for the matrix model expectation values (which are clearly different in general from the expectation values in the gauge theory). The parameters s_I introduced in (2.32) are thus the same as the filling fractions (2.17) of the matrix model.

The generating functions $\mathcal{G}_f^{J'}$ and \mathcal{R} are also meromorphic functions on the curve (2.31). We shall use the gauge theory resolvent \mathcal{R} extensively in the following, so let us describe its structure in details. Let us introduce

$$\psi_I(z) = \frac{\partial \mathcal{S}(z)}{\partial s_I}. \quad (2.35)$$

It is straightforward to show that the $\psi_I dz$ form a basis of log-normalizable holomorphic one-forms on \mathcal{C} ,

$$\psi_I = \frac{p_I}{y}. \quad (2.36)$$

The polynomials $p_I = z^{\tilde{d}-1} + \dots$ are fixed by the constraints

$$\frac{1}{2i\pi} \oint_{\alpha_J} \psi_I dz = \delta_{IJ}. \quad (2.37)$$

The gauge theory resolvent is then given by

$$\mathcal{R}(z) = \sum_I N_I \psi_I(z) + g_d(z), \tag{2.38}$$

where g_d is the disk matrix model resolvent (obtained by summing disk diagrams only),

$$g_d(z) = \left\langle\left\langle \text{Tr}_F \frac{1}{z - X} \right\rangle\right\rangle_{\text{disk}}. \tag{2.39}$$

If we define the “classical” sheet of the Riemann surface (2.31) by the condition

$$\mathcal{R}(z) \underset{z \rightarrow \infty}{\sim} \frac{N}{z}, \tag{2.40}$$

then \mathcal{R} has poles on the classical sheet at $z = b_Q$ with residue ν_Q if $\nu_Q = 1$, and poles on the second sheet at $z = b_Q$ with residue $1 - \nu_Q$ if $\nu_Q = 0$. We also have the useful identities

$$\frac{1}{2i\pi} \oint_{\alpha_I} \mathcal{R}(z) dz = N_I. \tag{2.41}$$

For completeness, let us finally mention that

$$\mathcal{G}_f^{f'}(z) = \left\langle\left\langle {}^T \tilde{Q}^{f'} \frac{1}{z - X} Q_f \right\rangle\right\rangle_{\text{disk}}. \tag{2.42}$$

The above equations determine all the $\langle \text{Tr}_F X^k \rangle$, $\langle W^\alpha W_\alpha \text{Tr}_F X^k \rangle$ and $\langle {}^T \tilde{Q}^{f'} X^q Q_f \rangle$ in terms of the s_I . Note that taking into account (2.35), (2.38) and (2.42) are perfectly consistent with (2.19) and (2.18).

2.4 Another formulation of the quantum equations of motion

A last important technical result that we need is an elegant reformulation of the quantum equations of motion (1.1) as constraints on the period integrals of $\mathcal{R} dz$ [12]. Using the open contours β_I defined in figure 1, and with the understanding that μ_0 is taken to infinity, the equations (1.1) are equivalent to

$$\frac{1}{2i\pi} \int_{\beta_I} \mathcal{R}(z) dz = \frac{1}{2i\pi} \ln \frac{U_0 \Lambda^{2N - N_f}}{\mu_0^{2N - \ell}} + k_I, \tag{2.43}$$

where the $k_{I\mathcal{S}}$ are integers defined modulo N_I . In particular, the periods on the compact cycles $\gamma_{IJ} = \beta_I - \beta_J$ are integers,

$$\frac{1}{2i\pi} \oint_{\gamma_{IJ}} \mathcal{R}(z) dz = k_I - k_J \in \mathbb{Z}. \tag{2.44}$$

Since this result is crucial for our purposes, we now present a short proof. The starting point is the so-called special geometry relations of the matrix model,

$$\frac{\partial^2 \mathcal{F}_s}{\partial s_I \partial s_J} = \int_{\beta_I} \psi_J dx + 2 \ln \frac{\mu_0}{\Lambda}, \tag{2.45}$$

as well as similar equations for the derivative of the disk partition function,

$$\frac{\partial \mathcal{F}_d}{\partial s_I} = \int_{\beta_I} g_d dx + \ln(\mu_0^{2\sum_{Q=1}^{\ell} \nu_Q - \ell} / U_0). \tag{2.46}$$

The details on the derivation of these matrix model relations can be found for example in [18]. Equations (2.18) and (2.38) then yield

$$\frac{\partial W_{DV}}{\partial s_I} = - \int_{\beta_I} \mathcal{R}(z) dz - \ln \frac{\mu_0^{2N-\ell}}{U_0 \Lambda^{2N-N_f}}. \tag{2.47}$$

This would correspond to a Veneziano-Yankielowicz term $-N_I \ln s_I$ in $\partial W_{DV} / \partial s_I$, whereas the correct term is $-\ln s_I^{N_I}$ as in (2.24). Taking into account this subtlety, we obtain (2.43). The k_I are defined modulo N_I because the contours β_I are defined modulo an integer multiple of α_I (the transformation $\beta_I \rightarrow \beta_I + \alpha_I$ is induced by continuously permuting a_I^- and a_I^+ , which obviously does not change the curve \mathcal{C} nor any gauge theory correlator, and thus yields the same solution).

The most natural form of (2.43) is

$$\mu_0^{2N-\ell} \exp \int_{\beta_I} \mathcal{R}(z) dz = U_0 \Lambda^{2N-N_f}. \tag{2.48}$$

From the above discussion, we know that there are automatically $\prod_{I=1}^{\tilde{d}} N_I$ distinct solutions to these equations, labelled by the \mathbb{Z}_{N_I} -valued integers k_I .

2.5 Discussion

The Konishi anomaly equations can be generalized to a very large class of models and put very powerful constraints on the set of chiral operators expectation values. However, any choice of the parameters s_I yields a perfectly sensible solution of the equations. For instance, the choice $s_I = 0$ for all I corresponds to no quantum corrections at all to W_{low} . Clearly, the anomaly equations do not address directly the problem of the strongly coupled gauge dynamics. This was explained in details already in the original paper [3]. Nevertheless, these equations *do* provide some important indirect information, since we know that the true answer must correspond to *some* particular values of the s_I . In particular, we know that \mathcal{S} and \mathcal{R} are functions defined on the two-sheeted surface (2.31), since this is true for *all* values of the s_I . The knowledge of this analytic structure will be of great help in the following.

Unlike the Konishi anomaly equations, the strong part of the Dijkgraaf-Vafa conjecture deals directly with genuine non-perturbative effects in the gauge theory, including infinite series of (fractional) instanton contributions and chiral symmetry breaking. It is also closely related to the problems of the mass gap and confinement in $\mathcal{N} = 1$ theories, as we shall review in section 5.

The original formulation of the conjecture in terms of the extremization of the function W_{DV} suggests that W_{DV} is a low energy effective superpotential and the equations (1.1) purely dynamical. It would clearly be very difficult to devise a proof along these lines.

For example, there is no convincing argument showing that the glueball fields S_I are the good low energy variables. Actually, this interpretation is logically disconnected from the Dijkgraaf-Vafa conjecture, and it is even unclear whether it is fully correct or not.

The equations (2.43) or (2.48), even though they are mathematically strictly equivalent to the original Dijkgraaf-Vafa formulation in terms of a superpotential, are beautifully simple and offer an attractive alternative starting point. At the very least, they suggest an entirely different interpretation. In particular, the equations (2.44) look like simple quantization conditions, which are often derived from general consistency constraints. We show in the next section that this is indeed the case, (2.44) being a consequence of gauge invariance. The full set of equations (2.48) then follows from an additional constraint, which is a simple Ward identity that we derive in section 4.

The relation between (2.44) and gauge invariance is not totally unexpected. *Assuming* that confinement is realized in the $U(N_I)$ factors of the low energy gauge group through the condensation of dyons of unit magnetic charge (the usual 't Hooft's mechanism), one can show that the integers k_I in (2.43) coincide with the electric charges of the dyons [11]. The Dirac-Schwinger-Zwanziger quantization condition, which is well-known to be a direct consequence of gauge invariance, is then equivalent to (2.44). Our main task in section 3 is to present a simple derivation of the same result that is free from any assumption on the low energy behaviour of the theory.

2.6 Other models

An important property of (2.44) is its universality. Even though the details of the Dijkgraaf-Vafa prescription (matrix model, definition of W_{DV}) are model-dependent, *the extremization of the Dijkgraaf-Vafa superpotential always yields quantization conditions of the form (2.44), with \mathcal{R} being defined as in (2.25) to be the resolvent associated with any adjoint chiral superfield in the theory.* This fact will find a simple explanation in the next section, but we wish to illustrate it using elementary methods on a representative example.

Let us introduce the theory with two adjoints X and Y and a tree-level superpotential of the form

$$W_{\text{tree}}(X, Y) = V_1(X) + V_2(Y) - \mu XY, \quad V_1' = P, \quad V_2' = Q. \quad (2.49)$$

This case is particularly interesting, because there are now two natural gauge theory resolvents,

$$\mathcal{R}^X(z) = \left\langle \text{Tr}_F \frac{1}{z - X} \right\rangle, \quad \mathcal{R}^Y(z) = \left\langle \text{Tr}_F \frac{1}{z - Y} \right\rangle, \quad (2.50)$$

for which constraints of the form (2.48) can be expected to be valid. This is non-trivial, because the constraints are then redundant, and not obviously consistent with each other.

The matrix model we have to consider is

$$e^{n^2 \mathcal{F}/s^2} = \mathcal{N} \int dX dY e^{-\frac{n}{s} \text{Tr}_F W_{\text{tree}}(X, Y)}, \quad (2.51)$$

with a Dijkgraaf-Vafa superpotential

$$W_{DV} = - \sum_I N_I \frac{\partial \mathcal{F}_s}{\partial s_I}. \quad (2.52)$$

It is then possible to show that, provided the normalization constant is chosen to be

$$\mathcal{N} \propto \Lambda^{-n^2/2} s^{-n^2/2}, \quad (2.53)$$

the equations of motion (1.1) take a form similar to (2.48). In particular, the quantization conditions

$$\frac{1}{2i\pi} \oint_{\gamma_{IJ}^X} \mathcal{R}^X(z) dz \in \mathbb{Z} \quad (2.54)$$

and

$$\frac{1}{2i\pi} \oint_{\gamma_{IJ}^Y} \mathcal{R}^Y(z) dz \in \mathbb{Z} \quad (2.55)$$

turn out to be equivalent to each other. The choice (2.53) for \mathcal{N} is also necessary for the standard interpretation in terms of Veneziano-Yankielowick terms to be valid. The contours γ_{IJ}^X and γ_{IJ}^Y are similar to the contours γ_{IJ} of figure 1, but with the γ_{IJ}^X (respectively γ_{IJ}^Y) going through the cuts of the resolvent \mathcal{R}^X (\mathcal{R}^Y). The proof of (2.54) and (2.55) uses the generalized Konishi anomaly equations (they have not appeared in the literature for the model (2.49), but they are straightforward to derive) and the special geometry relations for the matrix model that are derived in [6]. Full details are given in [18].

3. The quantization conditions

3.1 Presentation of the problem

Let M be an arbitrary adjoint chiral superfield. For example, in the model (2.3), we may consider any power of the elementary field X ; in the model (2.49), any combination of the form $\prod_k X^{p_k} Y^{q_k}$ would qualify. The resolvent for M is defined by

$$\mathcal{R}(z) = \left\langle \text{Tr}_F \frac{1}{z - M} \right\rangle. \quad (3.1)$$

The classical resolvent has the form

$$\mathcal{R}_{\text{cl}}(z) = \sum_{I=1}^D \frac{N_I}{z - a_I}, \quad (3.2)$$

where the integers N_I , $\sum_I N_I = N$, give the number of eigenvalues of M that are equal classically to a_I . Quantum mechanically, we assume that \mathcal{R} is a multi-sheeted analytic function. On the classical sheet, which is defined by the property that the large z asymptotics is the same as in the classical theory,

$$\mathcal{R}(z) \underset{z \rightarrow \infty}{\sim} \frac{N}{z}, \quad (3.3)$$

some of the poles (say the a_I for $1 \leq I \leq d$) in (3.2) are replaced by square root branch cuts, and the others (the a_Q for $Q \geq d+1$) remain poles that may be shifted from their classical value a_Q to a quantum value b_Q . The analytic structure is thus similar to the one given in figure 1 (in general there could be additional branch cuts on the second sheet, and the classical sheet could be glued to many different sheets by the square root branch cuts;

we shall work with an analytic structure of the form given by figure 1, because the other cases amount to straightforward generalizations). As explained in section 2, the above assumptions on \mathcal{R} can be derived from the generalized Konishi anomalies in many cases (including the cases that are relevant for our purposes), and so they are not restrictive.

The classical limit (3.2) and the analytic structure of \mathcal{R} imply trivially that

$$\frac{1}{2i\pi} \oint_{\alpha_I} \mathcal{R}(z) dz = N_I \in \mathbb{Z}. \tag{3.4}$$

We want to prove that gauge invariance implies that the other period integrals of $\mathcal{R}dz$ must be quantized as well,

$$\frac{1}{2i\pi} \oint_{\gamma_{IJ}} \mathcal{R}(z) dz \in \mathbb{Z}. \tag{3.5}$$

3.2 On gauge invariance

In [3], the single-trace gauge invariant chiral operators in the theory with no flavors were classified (a similar classification can be achieved in many other models). The full set is given by $\text{Tr}_F X^k$, $\text{Tr}_F X^k W^\alpha$ and $\text{Tr}_F X^k W^\alpha W_\alpha$. We call the ring generated by these elements the reduced chiral ring of the theory. Because of the factorization of chiral operators correlation functions, the structure of this ring is fully determined by the knowledge of the functions $\mathcal{R}(z)$ and $\mathcal{S}(z)$ defined in (2.25) and (2.26).

The most general element of the reduced chiral ring can be written as a *finite* sum of *finite* products of single trace chiral operators. This does not exhaust the full set of interesting chiral operators. In particular, $\mathcal{R}(z)$ and $\mathcal{S}(z)$ themselves are not in this ring. More generally, it is natural to consider operators of the form

$$O(z) = \sum_{k \geq K} O_k z^{-k}, \tag{3.6}$$

where the O_k are elements of the reduced chiral ring. The series defining the expectation values

$$\mathcal{O}(z) = \sum_{k \geq K} \langle O_k \rangle z^{-k} \tag{3.7}$$

are assumed to have a non-zero radius of convergence around $z = \infty$. Operators like (3.6) are usually introduced because they are generating functions for the $\langle O_k \rangle$. This interpretation comes from the expansion at large z . In particular, when z is in the disk on convergence, there is little to learn in considering $O(z)$ instead of the individuals O_k . However, we shall see that considering a priori $O(z)$ for any value of z introduces some interesting subtleties.

When z is not in the disk of convergence, $\mathcal{O}(z)$ is defined by analytic continuation. The result is in general a multivalued function on the z -plane. This multivaluedness is not obviously consistent with gauge invariance. Gauge invariance is manifest in an expansion like (3.6), but this expansion is only valid on a single sheet, and in general a similar expansion does not exist on the other sheets. To understand how the problem can arise, let us focus on the case where $\mathcal{O}(z)$ is built entirely in terms of a single adjoint field M in

the theory. The operators O_k are then polynomials in the $\text{Tr}_F M^q$, and thus $\mathcal{O}(z)$ depends only on the eigenvalues m_i of M on the sheet where (3.7) is valid, and thus on all the other sheets by analytic continuation,

$$\mathcal{O}(z) = \mathcal{O}(z; m_1, \dots, m_N). \quad (3.8)$$

The fundamental constraint from gauge invariance is then that $\mathcal{O}(z; m_i)$ *must be a symmetric function of the m_i* . If $\sigma \cdot$ is the operator implementing the gauge transformation associated with the permutation $\sigma \in S_N \subset U(N)$ of the eigenvalues, we must have

$$(\sigma \cdot \mathcal{O})(z; m_i) = \mathcal{O}(z; m_{\sigma(i)}) = \mathcal{O}(z; m_i). \quad (3.9)$$

This constraint is obviously satisfied when the series (3.7) converges, but is *not* insured for the analytic continuations.

To illustrate this point, let us consider an extremely simple classical toy example,

$$O(z) = \left(\text{Tr}_F \sqrt{z - M} \right)^2 = N^2 z - N \text{Tr}_F M + \frac{1}{4z} \left((\text{Tr}_F M)^2 - N \text{Tr}_F M^2 \right) + \dots \quad (3.10)$$

In terms of the eigenvalues, we have

$$\mathcal{O}(z; m_i) = \sum_{i,j} \sqrt{(z - m_i)(z - m_j)}. \quad (3.11)$$

The function $\mathcal{O}(z; m_i)$ has square root branch cuts that join pairs of eigenvalues. Let us start from a value of z for which the series in the right hand side of (3.10) converges, and let us consider the analytic continuation through the cut between m_{i_0} and m_{j_0} . This analytic continuation amounts to changing the sign of the term $\sqrt{(z - m_{i_0})(z - m_{j_0})}$ in (3.11), and is thus clearly inconsistent with the symmetry of \mathcal{O} under the permutation of m_{i_0} (or m_{j_0}) with any other eigenvalue m_i when $i \notin \{i_0, j_0\}$. This shows that $O(z)$ defined by (3.10) is not a good gauge invariant operator for all values of z . We could of course compute it at large z in the sheet where it is gauge invariant, and use it as a convenient book-keeping device for the $\text{Tr}_F X^k$. However, considering $(\text{Tr}_F \sqrt{z - M})^2$ for all values of z is inconsistent with gauge invariance.

In the general case, let us label by an index $a \geq 0$ the different sheets of the analytic function \mathcal{O} , and write $\mathcal{O}_a(z; m_i)$ its value on the a^{th} sheet. Suppose that the 0^{th} sheet contains the disk of convergence of (3.7), and thus in particular that

$$(\sigma \cdot \mathcal{O}_0)(z; m_i) = \mathcal{O}_0(z; m_i). \quad (3.12)$$

If the action of the operator $\sigma \cdot$ is smooth, the same equation must automatically be valid on any other sheet, by analytic continuation. This shows that if there exists a sheet, say number a , for which the permutation symmetry is violated,

$$(\sigma \cdot \mathcal{O}_a)(z; m_i) \neq \mathcal{O}_a(z; m_i), \quad (3.13)$$

then the gauge transformation $\sigma \cdot$ must be singular when acting on $\mathcal{O}(z)$ for some values of z . For example, if the 0^{th} and a^{th} sheets are joined by a branch cut, then $\sigma \cdot$ is clearly not well defined (two-valued) on this cut. This inconsistency is similar to the problems that appear in the presence of a Dirac string when the Dirac quantization condition is not satisfied.

3.3 The quantum characteristic function

3.3.1 Generalities

Let us introduce the characteristic function of M ,

$$\mathcal{F}(z) = \langle \det(z - M) \rangle, \tag{3.14}$$

with the basic property that

$$\frac{\mathcal{F}'(z)}{\mathcal{F}(z)} = \mathcal{R}(z). \tag{3.15}$$

By using

$$\det(z - M) = z^N e^{\text{Tr}_F \ln(1 - M/z)}, \tag{3.16}$$

we can obtain an expansion of the form (3.6),

$$\det(z - M) = z^N - \sum_{k \geq 1} F_k z^{N-k}, \tag{3.17}$$

where the elements of the reduced chiral ring $F_k = \frac{1}{k} \text{Tr}_F M^k + \dots$ are homogeneous polynomials in the $\text{Tr}_F M^q$ of degree k (M being of degree one) whose form does not depend explicitly on the number of colors N . For example,

$$\begin{aligned} F_1 &= \text{Tr}_F M, \\ F_2 &= \frac{1}{2} \text{Tr}_F M^2 - \frac{1}{2} (\text{Tr}_F M)^2, \\ F_3 &= \frac{1}{3} \text{Tr}_F M^3 - \frac{1}{2} (\text{Tr}_F M) (\text{Tr}_F M^2) + \frac{1}{6} (\text{Tr}_F M)^3, \text{ etc.} \end{aligned} \tag{3.18}$$

Picking a particular value of N amounts to imposing relations in the chiral ring,

$$F_{N+q} = 0, \quad q \geq 1, \tag{3.19}$$

that are equivalent to the fact that classically $\det(z - M)$ is a polynomial of degree N . Using (3.19) recursively, one obtains the standard identities

$$\text{Tr}_F M^q = P_{q,\text{cl}}(\text{Tr}_F M, \dots, \text{Tr}_F M^N), \quad q \geq 1, \tag{3.20}$$

where the $P_{q,\text{cl}}$ are homogeneous polynomials of degree q depending only on the N independent Casimir $\text{Tr}_F M^k$, $1 \leq k \leq N$, of $U(N)$.

The chiral ring relations (3.19) or (3.20) can get non-perturbative corrections,

$$\text{Tr}_F M^q = P_q(\text{Tr}_F M, \dots, \text{Tr}_F M^N; \Lambda). \tag{3.21}$$

The generalized Konishi anomaly equations put strong constraints on the form of the polynomials P_q . In particular, if the filling fractions s_I are not all equal to zero, the quantum corrections must be non-trivial. The equations (3.21) then imply that the quantum characteristic function (3.14) or (3.17) is not a polynomial, but an infinite series, the non-polynomial terms precisely generating the quantum corrections to the chiral ring.

3.3.2 The integral representation

The characteristic function $\mathcal{F}(z)$ is thus multivalued (this follows from (3.15) and the multivaluedness of $\mathcal{R}(z)$). The fact that a correlator can be multivalued in a field theory is of course not an inconsistency, and actually is a very common phenomenon. For example, it happens when the theory has different branches, or loosely speaking different phases. More concretely, when we consider the path integral for a theory with a (discrete or continuous) moduli space of vacua, this path integral is inherently multivalued, the ambiguity coming from the ambiguity in the choice of vacuum. It is also well-known that, in some cases, the different branches can be continuously related to each other by analytic continuation [9–12]. We are going to explain that the multivaluedness of $\mathcal{F}(z)$ can be understood in similar terms. Note however that there is an important difference: in cases like $\mathcal{F}(z)$, for which z enters the path integral only through the insertion of a local operator, the boundary conditions at infinity cannot depend on z . This means that the gauge theory vacuum (2.8) must be the same for all branches of \mathcal{F} .

The characteristic function can be expressed, *in a manifestly gauge invariant way for all values of z* , as an integral over anticommuting variables,

$$\det(z - M) = z^N \int d\psi d\chi \exp\left[\chi_a (\delta_b^a - z^{-1} M_b^a) \psi^b\right]. \quad (3.22)$$

Strictly speaking, this formula is only valid classically. Quantum mechanically, a smearing in space-time is necessary, and the integral in the right hand side of (3.22) is to be interpreted as being a *path* integral localized around a certain space-time point x , in the limit where the smearing distribution goes to the Dirac δ . This smearing is very similar in nature to the smearing of the glueball field introduced in [19]. In particular, $(\chi_a M_b^a \psi^b)^k$ can be non-zero for arbitrary powers k . When expanded around $\psi^a = \chi_a = 0$, the integral (3.22) then yields an infinite series in z . Using Wick’s theorem and the contraction $\overline{\chi_a \psi^b} = \delta_a^b$ in the limit where the smearing is removed, it goes to the expansion (3.17).

Expanding around $\psi^a = \chi_a = 0$ is not the only possibility. Indeed, the classical equations of motion for (3.22) read

$$M_b^a \psi^b = z \psi^a, \quad \chi_a M_b^a = z \chi_b. \quad (3.23)$$

When z coincides with an eigenvalue of M , there is a moduli space of solutions (“Higgs branch”) parametrized by the gauge invariant “meson”

$$\sigma = \chi_a \psi^a. \quad (3.24)$$

This is a hint that (3.22) can indeed be multivalued. Quantum mechanically, the eigenvalues of M fluctuate. On the one hand, these fluctuations can lift the flat direction in σ , and on the other hand the “Higgs branch” can exist for any z , with a given value of $\langle \sigma \rangle = \sigma_H(z)$. This behaviour is reproduced for example by a simple toy model for which the quantum corrections are modeled by a gaussian integration over M (note however that the gauge theory fluctuations are much more subtle: single trace operators behave as if the eigenvalues were fluctuating, whereas multi-trace correlators factorize as if they were dominated by a non-fluctuating master field).

Taking the derivative with respect to z , the formula (3.22) shows that the condensate $\langle\sigma\rangle$ is nothing but the gauge theory resolvent \mathcal{R} ,

$$\langle\sigma\rangle(z) = \mathcal{R}(z). \tag{3.25}$$

We thus have a nice interpretation of the multivaluedness of $\mathcal{R}(z)$, that correspond to the different “phases” of the integral (3.22).

3.3.3 A simple proof of the quantization conditions

The above interpretation may be innocent-looking, but it has deep consequences. The path integral (3.22) can be multivalued because it can have several “vacua.” In particular, once we pick a vacuum, all the ambiguity must be lifted. But the vacua are characterized by the set of gauge invariant expectation values that can be built from the elementary fields over which we integrate. In the case of (3.22), all we have is the gauge invariant meson (3.24), obtained by taking the derivative with respect to z , or higher powers of σ , obtained by taking higher derivatives. This implies that, even though (3.22) may be a multivalued function of z , it must have precisely the same sheets and branch cuts as $\langle\sigma\rangle(z) = \mathcal{R}(z)$. In other words, it must be a meromorphic function on the Riemann surface (2.31).

Let us analyze the consequences of this fact, focusing on the case of the theory (2.3) for concreteness. By integrating the relation (3.15) from the point at infinity on the classical sheet to z , we obtain

$$\mathcal{F}(z) = \mu_0^N \exp \int_{\mu_0}^z \mathcal{R}(z') dz'. \tag{3.26}$$

This formula shows that \mathcal{F} is generically a multivalued function on the curve (2.31), because one must specify the contour from the point at infinity μ_0 to z to do the integral. A first simplification occurs thanks to the conditions (3.4). We can define unambiguously $\mathcal{F}(z)$ on the classical sheet by integrating along any path that remains on this sheet. The answer does not depend on how many times the path circles around the branch cuts of \mathcal{R} . This means that \mathcal{R} and \mathcal{F} have the same branch cuts on the classical sheet, and the latter can be equally well characterized by the properties (3.3) or by

$$\mathcal{F}(z) \underset{z \rightarrow \infty}{\sim} z^N. \tag{3.27}$$

We can say more about the classical sheet branch cuts of \mathcal{F} : as for \mathcal{R} , they are square root branch cuts. This is so because going twice across the cut corresponds to going along a contour whose homotopy class is trivial. Let us also note that the poles of \mathcal{R} become simple zeros of \mathcal{F} . Complications arise when we consider the analytic continuations across the classical sheet branch cuts. Unlike for \mathcal{R} , if we go through two different branch cuts, we obtain a priori two inequivalent continuations of \mathcal{F} . Equation (3.26) shows that this does not occur if and only if

$$\exp \oint_{\gamma_{IJ}} \mathcal{R}(z) dz = 1, \tag{3.28}$$

in which case \mathcal{F} and \mathcal{R} are defined on the same surface (2.31) as required. Obviously, (3.28) is equivalent to the quantization conditions (3.5).

We are now going to show that the same result also follows from a rigorous implementation of the constraints (3.9) on \mathcal{F} .

3.4 The permutation symmetry

3.4.1 Dependence of the eigenvalues

We now focus on the model (2.3) with $M = X$ for concreteness. Let us go to a gauge where X is diagonal (this is the supersymmetric version of 't Hooft's abelian projection [20]),

$$X = \text{diag}(x_1, \dots, x_N). \tag{3.29}$$

The gauge fixing procedure in the gauge theory, which mimics the gauge fixing in the associated matrix model (2.13), was discussed in [21]. The eigenvalues x_I are operators in the quantum theory. In a given vacuum (2.8), we expand each eigenvalue around its classical value. For example, we may choose

$$x_i = a_I + \hat{x}_i \quad \text{for} \quad \sum_{J < I} N_J + 1 \leq i \leq \sum_{J \leq I} N_J. \tag{3.30}$$

The gauge theory resolvent can be decomposed in a natural way as

$$\mathcal{R}(z) = \sum_{I=1}^{\tilde{d}} N_I \mathcal{R}_I(z) + \sum_{Q=1}^{\ell} \frac{\nu_Q}{z - b_Q} \tag{3.31}$$

with

$$\mathcal{R}_I(z) = \frac{1}{N_I} \oint_{\alpha_I} \frac{dz'}{2i\pi} \frac{\mathcal{R}(z')}{z - z'} = \left\langle \frac{1}{z - x_i} \right\rangle, \quad \sum_{J < I} N_J + 1 \leq i \leq \sum_{J \leq I} N_J. \tag{3.32}$$

The variable z in (3.32) is taken to be outside the contour α_I on the classical sheet (see figure 1). Using (3.4), we see that the function \mathcal{R}_I has only one square root branch cut $[a_I^-, a_I^+]$ on the classical sheet. The decomposition (3.31) yields a similar decomposition of the characteristic function,

$$\mathcal{F}(z) = \prod_{I=1}^{\tilde{d}} \mathcal{F}_I(z)^{N_I} \prod_{Q=1}^{\ell} (z - b_Q)^{\nu_Q}, \tag{3.33}$$

with

$$\mathcal{F}_I(z) = \mu_0 \exp \int_{\mu_0}^z \mathcal{R}_I(z') dz'. \tag{3.34}$$

Equation (3.33) is the quantum version of the classical formula

$$\mathcal{F}_{\text{cl}}(z) = \prod_{I=1}^{\tilde{d}} (z - a_I)^{N_I} \prod_{Q=1}^{\ell} (z - b_Q)^{\nu_Q}. \tag{3.35}$$

The basic property of the expansion (3.31) is that each factor \mathcal{R}_I depends only on the eigenvalues x_i for $\sum_{J < I} N_J + 1 \leq i \leq \sum_{J \leq I} N_J$. The same property is true for the \mathcal{F}_I that are expressed by (3.34) in terms of \mathcal{R}_I only. More precisely, in the quantum theory, the classical eigenvalue x_i is replaced by the full set of correlators $\{x_i^k, k \geq 1\}$, or equivalently

by a probability distribution ρ_i . This probability distribution is the same for all i satisfying the condition $\sum_{J < I} N_J + 1 \leq i \leq \sum_{J \leq I} N_J$, and will be denoted by ρ_I . The resolvent \mathcal{R}_I depends only on ρ_I and can be expressed as

$$\mathcal{R}_I(z) = \mathcal{R}(z; \rho_I) = \int \frac{\rho_I(z') dz'}{z - z'}. \quad (3.36)$$

Conversely, the distribution ρ_I is given by the discontinuity across the classical sheet branch cut,

$$\rho_I(x) = \frac{i}{2\pi} (\mathcal{R}_I(x + i\epsilon) - \mathcal{R}_I(x - i\epsilon)), \quad x \in [a_I^-, a_I^+]. \quad (3.37)$$

A potentially confusing point² is that the expansions (2.38) and (3.31) of \mathcal{R} are similar-looking (at least for $N_f = 0$), and that the forms $\mathcal{R}_I dz$ and $\psi_I dz$ both satisfy

$$\frac{1}{2i\pi} \oint_{\alpha_J} \psi_I dz = \delta_{IJ}, \quad \frac{1}{2i\pi} \oint_{\alpha_J} \mathcal{R}_I dz = \delta_{IJ}. \quad (3.38)$$

These similarities are very misleading, and the expansions (2.38) and (3.31) are of *completely different origins*. In particular, the \mathcal{R}_I defined by (3.32) are not single-valued on the curve (2.31) (even though the sum (3.31) obviously is), they have only one branch cut $[a_I^-, a_I^+]$ on the classical sheet (the $[a_J^-, a_J^+]$ for $J \neq I$ are not branch cuts), and they have in general an *infinite* number of sheets (they are not algebraic functions). This is very different from the ψ_I , that are single-valued on (2.31), have branch cuts on $[a_J^-, a_J^+]$ for all J on the classical sheet, and satisfy a degree two algebraic equation. Moreover, for the forms $\psi_I dz$, the constraint (3.38) is the non-trivial statement of Poincaré duality. In particular, the fact that the α_J -period integral of $\psi_I dz$ is zero for $J \neq I$ is a non-trivial statement (it gives constraints on the polynomials p_I in (2.36)). On the other hand, the similar statement on $\mathcal{R}_I dz$ is *trivial*. It just follows from the fact that \mathcal{R}_I has no branch cut or other singularity on $[a_J^-, a_J^+]$ or inside α_J , and thus the contour can be deformed to a point when doing the corresponding integral.

The functions \mathcal{R}_I or \mathcal{F}_I are important for our purposes because they depend only on ρ_I , and thus the action of the permutation of the eigenvalues can be easily studied on them as we shall see below. This simple property is not shared by the ψ_I .

3.4.2 Permuting the eigenvalues

In the sake of simplicity, let us first consider the case where all the $N_I = 1$ and $\nu_Q = 0$ (this is relevant, for example, to the solution of the $\mathcal{N} = 2$ theories, see section 5). The generalized Konishi anomalies imply that the distributions ρ_I are parametrized by the branching points a_K^\pm of the curve (2.31) and the parameters b_Q (the latter enters the solution through the disk resolvent in (2.18)),

$$\mathcal{F}_I(z) = \mathcal{F}(z; \rho_I) = \mathcal{F}_I(z; a_K^\pm; b_Q). \quad (3.39)$$

The permutation operator $\sigma_{IJ} = \sigma_{JI}$ exchanges the I^{th} and J^{th} eigenvalues, that is to say exchanges the distributions ρ_I and ρ_J . In explicit formulas, it is straightforward to see from the solution described in section 2 that this amounts to exchanging the branching

²I thank the referee for raising this point.

points a_I^\pm and a_J^\pm , because we have

$$\begin{aligned} \rho_I(x; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) &= (\sigma_{IJ} \cdot \rho_J)(x; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) \\ &= \rho_J(x; \dots, a_J^\pm, \dots, a_I^\pm, \dots; b_Q). \end{aligned} \quad (3.40)$$

In particular, this shows that

$$\begin{aligned} \mathcal{F}_I(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) &= (\sigma_{IJ} \cdot \mathcal{F}_J)(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) \\ &= \mathcal{F}_J(z; \dots, a_J^\pm, \dots, a_I^\pm, \dots; b_Q), \end{aligned} \quad (3.41)$$

and

$$\begin{aligned} \mathcal{F}_K(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) &= (\sigma_{IJ} \cdot \mathcal{F}_K)(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) \\ &= \mathcal{F}_K(z; \dots, a_J^\pm, \dots, a_I^\pm, \dots; b_Q) \quad \text{for } K \notin \{I, J\}. \end{aligned} \quad (3.42)$$

These equations are of course valid on all sheets. If $\mathcal{F}_{I,0}(z)$ is the value of $\mathcal{F}_I(z)$ on the classical sheet, and $\mathcal{F}_{I,1}(z)$ the value on the second sheet that is glued to the classical sheet by the branch cut $[a_I^-, a_I^+]$ (\mathcal{F}_I has in general additional sheets, but they are irrelevant for our purposes), then

$$\mathcal{F}_{I,0}(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) = \mathcal{F}_{J,0}(z; \dots, a_J^\pm, \dots, a_I^\pm, \dots; b_Q), \quad (3.43)$$

$$\mathcal{F}_{I,1}(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) = \mathcal{F}_{J,1}(z; \dots, a_J^\pm, \dots, a_I^\pm, \dots; b_Q). \quad (3.44)$$

We can now implement the constraint (3.9) on \mathcal{F} . It is equivalent to the identity

$$\begin{aligned} \mathcal{F}(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) &= (\sigma_{IJ} \cdot \mathcal{F})(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) \\ &= \mathcal{F}(z; \dots, a_J^\pm, \dots, a_I^\pm, \dots; b_Q), \end{aligned} \quad (3.45)$$

which must be valid on all sheets.³ On the classical sheet, this is of course automatically satisfied by using (3.43),

$$\begin{aligned} \mathcal{F}_0(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) &= \prod_{K=1}^N \mathcal{F}_{K,0}(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) \\ &= \mathcal{F}_0(z; \dots, a_J^\pm, \dots, a_I^\pm, \dots; b_Q). \end{aligned} \quad (3.46)$$

Let us now consider the analytic continuation $\mathcal{F}_1^{(I)}$ of \mathcal{F}_0 through the I^{th} branch cut (a priori, $\mathcal{F}_1^{(I)}$ and $\mathcal{F}_1^{(J)}$ could be distinct functions if $I \neq J$). Remembering that $[a_I^-, a_I^+]$ is not a branch cut of the $\mathcal{F}_{K,0}$ for $K \neq I$, we find

$$\mathcal{F}_1^{(I)}(z) = \mathcal{F}_{I,1}(z) \prod_{K \neq I} \mathcal{F}_{K,0}(z). \quad (3.47)$$

³Let us stress that the action of σ_{IJ} does not mix up the different sheets. The a^{th} sheet can be characterized by the coefficients $c_{a,k}(\rho_I)$ of the large z expansion, which have a certain sheet-dependent functional dependence on the distribution ρ_I . Permuting ρ_I and ρ_J clearly does not change a .

The identity (3.45) then yields

$$\begin{aligned} \mathcal{F}_{I,1}(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) \prod_{K \neq I} \mathcal{F}_{K,0}(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) = \\ \mathcal{F}_{I,1}(z; \dots, a_J^\pm, \dots, a_I^\pm, \dots; b_Q) \prod_{K \neq I} \mathcal{F}_{K,0}(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q). \end{aligned} \quad (3.48)$$

Using (3.43) and (3.44) on the right hand side of this equation, we obtain

$$\begin{aligned} \mathcal{F}_{I,1}(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) \mathcal{F}_{J,0}(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) = \\ \mathcal{F}_{J,1}(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q) \mathcal{F}_{I,0}(z; \dots, a_I^\pm, \dots, a_J^\pm, \dots; b_Q). \end{aligned} \quad (3.49)$$

This identity simply tells us that the analytic continuation of \mathcal{F}_0 through the J^{th} branch cut,

$$\mathcal{F}_1^{(J)}(z) = \mathcal{F}_{J,1}(z) \prod_{K \neq J} \mathcal{F}_{K,0}(z), \quad (3.50)$$

must be equal to the analytic continuation (3.47) through the I^{th} branch cut. Using (3.26), this immediately yields the quantization condition (3.5).

The most general case of arbitrary N_I and ν_Q can be treated along similar lines. One must take into account that all the eigenvalues that are sitting at the same point classically are constrained by the generalized Konishi equations to have the same resolvent (3.32) quantum mechanically. Moreover, the eigenvalues at b_Q cannot fluctuate. The constraints on \mathcal{F} are then obtained by exchanging the N_I eigenvalues at a_I with the N_J eigenvalues at a_J . This is a symmetry as long as one exchanges N_I and N_J at the same time, corresponding to a combination of permutations that exchange the I^{th} and J^{th} blocks of the matrix X .

4. One more Ward identity

Taking into account (2.47), the quantization conditions (3.5) are equivalent to

$$\frac{\partial W_{\text{DV}}}{\partial s_I} = \frac{\partial W_{\text{DV}}}{\partial s_J}. \quad (4.1)$$

We can use these $\tilde{d} - 1$ equations to express the \tilde{d} parameters s_I in terms of a single parameter $s = \sum_I s_I$,

$$s_I = \hat{s}_I(s). \quad (4.2)$$

This parameter s is the last indeterminate in our problem. The equation (2.33) shows that it is identified with the glueball superfield (2.12) expectation value, $s = \langle S \rangle$. It is natural to introduce a new analytic function defined by

$$\hat{W}_{\text{DV}}(s) = W_{\text{DV}}(\hat{s}_I(s)). \quad (4.3)$$

To complete the proof of the Dijkgraaf-Vafa conjecture (1.1), we need to show that

$$\hat{W}'_{\text{DV}}(s = \langle S \rangle) = 0, \quad (4.4)$$

where the $'$ denotes the derivative with respect to s . Let us note that when $\tilde{d} = 1$ and the low energy gauge group has a single $U(N_1)$ factor, $\hat{W}_{\text{DV}} = W_{\text{DV}}$ and equation (4.4) is all we have. This simple “one-cut” case contains already a great deal of interesting non-perturbative physics [22, 9]. The aim of this section is to show that (4.4) follows from a simple Ward identity.

4.1 The quantum glueball superpotential

The simplest way to present the proof is to use the so-called quantum glueball superpotential $W(s)$. This is defined to be the Legendre transform of W_{low} , defined in (2.1), with respect to the gauge coupling constant. More precisely, in the theory with N_f flavors, the expectation value of the glueball superfield (2.12) is given by the equation (2.11) (with the trivial modification $\ln \Lambda^{2N-N_f} \rightarrow 2i\pi\tau$ when $N_f = 2N$). This equation can be inverted to find $\ln \Lambda^{2N-N_f}$ as a function f of $\langle S \rangle = s$. Indicating the dependence on the various couplings explicitly, with in particular (2.14), the glueball superpotential is then defined by

$$W(s; t_K, \ln \Lambda^{2N-N_f}) = W_{\text{low}}(t_K, f(s; t_K)) + (\ln \Lambda^{2N-N_f} - f(s; t_K))s. \quad (4.5)$$

By construction, $W(s)$ is linear in $\ln \Lambda^{2N-N_f}$, and satisfies the equations

$$\frac{\partial W}{\partial g_k} = \frac{1}{k+1} \langle \text{Tr}_F X^{k+1} \rangle, \quad \frac{\partial W}{\partial m_{f,k}^{f'}} = \langle {}^T \tilde{Q}^f X^k Q_{f'} \rangle, \quad (4.6)$$

$$W'(s = \langle S \rangle) = 0. \quad (4.7)$$

These equations are immediate consequences of the definition (4.5) and of (2.10) and (2.11). In particular, (4.7) implies that if we could prove that

$$W \stackrel{?}{=} \hat{W}_{\text{DV}}, \quad (4.8)$$

then the last Dijkgraaf-Vafa equation (4.4) would automatically follow. We are thus going to derive (4.8) in the following.

This derivation will necessitate a little bit of more work, but from what we already know we can immediately deduce that

$$W(s; t_K, \ln \Lambda^{2N-N_f}) = \hat{W}_{\text{DV}}(s; t_K, \ln \Lambda^{2N-N_f}) + h(s), \quad (4.9)$$

where h depends on s only (not on the couplings t_K or Λ) and $h(0) = 0$. This last condition on h follows from the fact that, by construction, W and \hat{W}_{DV} have the same classical, $s \rightarrow 0$, limit. Moreover, the dependence on $\ln \Lambda^{2N-N_f}$ is through the simple linear term $s \ln \Lambda^{2N-N_f}$ on both sides of (4.9). This is clear for W from the definition (4.5), and it is also true for \hat{W}_{DV} , that depends on Λ only through the normalization factor (2.23) in the matrix integral. Actually, the Λ -dependence of \mathcal{N} was chosen precisely for this matching to be valid. Finally, (4.1), (4.2) and (4.3) imply that $\partial_{t_K} W_{\text{DV}} = \partial_{t_K} \hat{W}_{\text{DV}}$. Then the equations (4.6) on the one hand, and (2.19) on the other hand, show that the t_K -dependence on both sides of (4.9) must be the same.

4.2 The last Dijkgraaf-Vafa equation

Thus we have to prove that $h = 0$, or more precisely that h must be a linear function, since a linear shift with a coupling-independent slope can be absorbed in the normalization \mathcal{N} (or equivalently in a rescaling of Λ on the matrix model side).

Our strategy is as follows. Let us introduce the first order differential operator

$$\mathcal{D} = s \frac{\partial}{\partial s} + \sum_k g_k \frac{\partial}{\partial g_k}. \quad (4.10)$$

We are first going to show that matrix model identities imply that

$$\mathcal{D} \cdot \hat{W}_{\text{DV}} = \hat{W}_{\text{DV}}. \quad (4.11)$$

This equation is just an identity that follows from the definition of \hat{W}_{DV} . It strongly depends in particular on the s -dependence of the normalization factor \mathcal{N} (2.23). Put in another way, we have chosen this s -dependence precisely for the equation (4.11) to be valid. In particular, it fixes uniquely the Veneziano-Yankielowicz contributions to \hat{W}_{DV} . Second, we are going to show that a gauge theory Ward identity implies that

$$\mathcal{D} \cdot W = W. \quad (4.12)$$

Combining (4.11), (4.12) and (4.9), we obtain

$$\mathcal{D} \cdot h = h = sh', \quad (4.13)$$

whose most general solution is a linear function, as announced.

4.2.1 Matrix model identities

The simplest way to derive (4.11) is to start from the matrix integral and to perform the infinitesimal variations

$$\delta s_I = \epsilon s_I, \quad \delta s = \epsilon s, \quad (4.14)$$

on both sides of (2.13). The matrix model vacuum (2.16) is left invariant, because (2.17) implies that the n_I are left invariant. By taking into account the s -dependence of the normalization factor \mathcal{N} given by (2.23), we obtain immediately

$$\begin{aligned} -2\mathcal{F} + \sum_I s_I \frac{\partial \mathcal{F}}{\partial s_I} &= \frac{s}{n} \langle\langle \text{Tr}_F W_{\text{tree}}(X) + {}^T \tilde{Q}^f m_f^{f'}(X) Q_{f'} \rangle\rangle - \frac{s^2 N_f}{n} \\ &= - \sum_k g_k \frac{\partial \mathcal{F}}{\partial g_k} + \frac{s}{n} \langle\langle {}^T \tilde{Q}^f m_f^{f'}(X) Q_{f'} \rangle\rangle - \frac{s^2 N_f}{n}. \end{aligned} \quad (4.15)$$

Expanding in powers of $1/n$ as in (2.15), and using

$$\langle\langle {}^T \tilde{Q}^f m_f^{f'}(X) Q_{f'} \rangle\rangle_{\text{disk}} = s N_f \quad (4.16)$$

which is a direct consequence of the matrix model loop equations (see for instance [17]), we obtain

$$\sum_I s_I \frac{\partial \mathcal{F}_s}{\partial s_I} + \sum_k g_k \frac{\partial \mathcal{F}_s}{\partial g_k} = 2\mathcal{F}_s, \tag{4.17}$$

$$\sum_I s_I \frac{\partial \mathcal{F}_d}{\partial s_I} + \sum_k g_k \frac{\partial \mathcal{F}_d}{\partial g_k} = \mathcal{F}_d. \tag{4.18}$$

Relations similar to (4.17) have been considered in special cases [23]. Combining (4.17) and (4.18) with (2.18), we get

$$\sum_I s_I \frac{\partial W_{\text{DV}}}{\partial s_I} + \sum_k g_k \frac{\partial W_{\text{DV}}}{\partial g_k} = W_{\text{DV}}, \tag{4.19}$$

and equation (4.11) then follows from the definition (4.3).

4.2.2 Gauge theory identities

The anomaly equations discussed in section 2.3 are generalizations of more standard (anomalous) Ward identities associated with the symmetries of the gauge theory (2.3). There is however a useful non-anomalous $U(1)_R$ symmetry that yields a new constraint. If we denote by θ the superspace coordinates, by λ the gluino, and w any effective superpotential, the assignment of charges under this $U(1)_R$ is as follows:

	θ	λ	X	Q_f	\tilde{Q}^f	g_k	$m_{f,q}^{f'}$	w	
$U(1)_R$	1	1	0	1	1	2	0	2	.

(4.20)

This symmetry implies that

$$W_{\text{low}} = \sum_k g_k \frac{\partial W_{\text{low}}}{\partial g_k} = \langle \text{Tr}_F W_{\text{tree}}(X) \rangle. \tag{4.21}$$

After Legendre transforming to W , we obtain (4.12), which completes the proof.

4.2.3 The theory with two flavors

The theory with two flavors can be treated along similar lines.

On the matrix model side, if we denote collectively by t_K the couplings in the tree-level superpotential (2.49), the planar partition function defined by (2.51) with the normalization factor \mathcal{N} given by (2.53) satisfies

$$\sum_I s_I \frac{\partial \mathcal{F}_s}{\partial s_I} + \sum_K t_K \frac{\partial \mathcal{F}_s}{\partial t_K} + \frac{s^2}{2} = 2\mathcal{F}_s. \tag{4.22}$$

This is the analogue of (4.17) for the model with fundamental flavors. If

$$\mathcal{D} = s \frac{\partial}{\partial s} + \sum_K t_K \frac{\partial}{\partial t_K}, \tag{4.23}$$

equation (4.22) yields

$$\mathcal{D} \cdot \hat{W}_{\text{DV}} - Ns = \hat{W}_{\text{DV}} \tag{4.24}$$

for the Dijkgraaf-Vafa superpotential defined by (2.52) and (4.3).

On the gauge theory side, we use the anomalous $U(1)_R$ symmetry

$$U(1)_R \begin{matrix} \theta & \lambda & X & Y & \Lambda^N & t_K & w \\ 1 & 1 & 0 & 0 & -2N & 2 & 2. \end{matrix} \tag{4.25}$$

to show that

$$W_{\text{low}} = \sum_K t_K \frac{\partial W_{\text{low}}}{\partial t_K} - N \frac{\partial W_{\text{low}}}{\partial \ln \Lambda^N} = \langle \text{Tr}_F W_{\text{tree}}(W, Y) \rangle - N \langle S \rangle. \tag{4.26}$$

Legendre transforming, we obtain

$$\mathcal{D} \cdot W - Ns = W. \tag{4.27}$$

Comparing this with (4.24), and using the analogue of (4.9), we conclude that $W = \hat{W}_{\text{DV}}$.

4.3 A note on the most general solutions to the constraints

There is one implicit assumption that we have made so far.⁴ If $N_I \neq 0$, we have taken for granted that the branch cut $[a_I^-, a_I^+]$ opens up, or equivalently that s_I is not identically zero. This is necessary to obtain the full set of quantization conditions (3.5). However, another possibility is to let some of the s_I to be identically zero (let's say for $1 \leq I \leq \delta$), even though the corresponding unbroken factors of the gauge group $U(N_I)$ are non-trivial. The remaining parameters s_K for $K \geq \delta + 1$ are then fully determined by the quantization conditions $\oint_{\gamma_{KL}} \mathcal{R}(z) dz \in 2i\pi\mathbb{Z}$ for $K, L > \delta$ that follow from the analysis of section 3 as well as from the additional constraint discussed above (for example (4.21)). Let us emphasize that the solutions so obtained are different from the $N_I = 0$ solutions (which also have $s_I = 0$), because the α_I -period of $\mathcal{R}(z)$ are constrained to be equal to N_I .

What are we supposed to do with these additional solutions? One consistent possibility is to consider that they are unphysical and disregard them (this is consistent because they cannot be obtained by analytic continuation from the other solutions: they belong to a different phase). It is however natural to try to interpret them, since they are good solutions of our set of constraints. Clearly they cannot correspond to the usual vacua (2.8), and thus if they are physical they imply the existence of new vacua, that we denote for reasons to become clear in a moment

$$|N_1, \text{KS}; \dots; N_\delta, \text{KS}; N_{\delta+1}, k_{\delta+1}; \dots; N_d, k_d; \nu_1; \dots; \nu_\ell\rangle. \tag{4.28}$$

For example, when the gauge group is unbroken, we would have, on top of the usual vacua $|N, k\rangle$, an additional vacuum $|N, \text{KS}\rangle$. In this vacuum, the expectation value of the glueball superfield is zero by definition,

$$\langle N, \text{KS} | S | N, \text{KS} \rangle = 0. \tag{4.29}$$

⁴I thank the referee for raising this point.

The state $|N, \text{KS}\rangle$ has all the required properties to be identified with the chirally symmetric vacuum postulated by Kovner and Shifman in [15]. The more general cases (4.28) are the immediate generalizations of the Kovner-Shifman vacua for unbroken $U(N_1) \times \cdots \times U(N_d)$ gauge group.

The existence of the Kovner-Shifman vacua is controversial and has never been rigorously established. It is interesting to find that they naturally fit in the framework of the present paper, and that the most general consistent solution actually corresponds to the general states of the form (4.28). For this reason, and also because, to my knowledge, a convincing argument that rules them out has not appeared, we believe that these hypothetical new phases of $\mathcal{N} = 1$ gauge theories deserve further investigations. Let us stress however that a very plausible possibility is that there exists additional constraints that restrict the physical vacuum states to be of the standard form (2.8).

The original Kovner-Shifman vacua [15], if they exist, must belong to the strongly coupled regime of $\mathcal{N} = 1$ gauge theories, and this makes their study particularly difficult. However, we want to point out that some of our generalized KS vacua (4.28) can be made arbitrarily weakly coupled by adjusting the parameters. This occurs when the unbroken gauge group has only abelian factors. For example, this is the case for a Coulomb branch with symmetry breaking pattern $U(N) \rightarrow U(1)^N$, which is also relevant to $\mathcal{N} = 2$ super Yang-Mills [7]. Thus in principle we may have KS vacua in $\mathcal{N} = 2$ as well! The existence of weakly coupled KS vacua is particularly puzzling, because explicit instanton calculations are then reliable and seem to rule them out. There might remain the possibility that some subtleties of the semiclassical path integral have been overlooked.

5. The confinement and mass gap properties

In this last section, we are going to apply the above results to the problem of the creation of a mass gap and of color confinement in $\mathcal{N} = 1$ super Yang-Mills theory. The strategy we use to prove confinement is to deform a $\mathcal{N} = 2$ theory by adding a tree-level superpotential, as explained by Seiberg and Witten in [14]. Our contribution is to waive the two main non-trivial assumptions on which this famous reasoning is based.

The first assumption is that the solution of the $\mathcal{N} = 2$ theory given in [14], and its generalizations, are correct. This can now be derived, using the results of sections 2–4, following the approach proposed in [7]. Many examples have been studied in the literature, for instance in [24]. However, the particularly interesting case of the $N_f = 2N$ theory, for which the Yang-Mills coupling does not run, doesn't seem to have been published. Moreover, even for the cases with $N_f < 2N$, which can be obtained from the $N_f = 2N$ theory by integrating out the quarks, the analysis presented in the literature is often incomplete, because the last Dijkgraaf-Vafa equation (4.4) is usually not implemented. For these reasons, and also because the result is instructive, we briefly present the derivation for the $N_f = 2N$ theory in the next subsection.

The second and most important assumption is that the $\mathcal{N} = 1$ theory has a mass gap. This is explained in section 5.2, where we show how this fundamental property can actually be derived by combining the solutions of both the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ theories obtained previously.

5.1 The $\mathcal{N} = 2$ theory with $N_f = 2N$

We consider the theory (2.3) with a degree $d = N + 1$ tree-level superpotential parametrized as

$$W'_{\text{tree}}(x) = \epsilon P_N(x) = \epsilon \prod_{I=1}^N (x - a_I), \quad (5.1)$$

polynomials $m_f^{f'}$ of the form

$$m_f^{f'}(X) = (X - m_f) \delta_f^{f'}, \quad (5.2)$$

and we focus on the vacuum $|N_I = 1; \nu_Q = 0\rangle$ which has $\tilde{d} = d = N$ and gauge symmetry breaking $U(N) \rightarrow U(1)^N$. The Riemann surface (2.31) takes the form

$$C : y^2 = P_N^2 - p_{N-1}(x), \quad (5.3)$$

for a certain polynomial p_{N-1} of degree $\deg p_{N-1} \leq N - 1$.

The N unknown coefficients in p_{N-1} are determined by the N quantum equations of motion (2.44) and (4.4). As we have explained in details, (2.44) is equivalent to the fact that the characteristic function \mathcal{F} is a meromorphic function on the curve (5.3). It has two poles of order N at infinity on the classical and second sheets, and $2N$ simple zeros at $z = m_f$ on the second sheet. Moreover, equation (4.4), or equivalently one of the equation (2.48) (with $\ell = 2N$, $U_0 = 1$ and Λ^{2N-N_f} replaced by $e^{2i\pi\tau}$ since we are in the theory with zero β function) is equivalent to a simple asymptotic condition on \mathcal{F} . If we note, as in section 3, $\mathcal{F}_0(z)$ the value of \mathcal{F} on the classical sheet and $\mathcal{F}_1(z)$ the value on the second sheet, then we have

$$\lim_{\mu_0 \rightarrow \infty} \frac{\mathcal{F}_1(\mu_0)}{\mathcal{F}_0(\mu_0)} = e^{2i\pi\tau} = q. \quad (5.4)$$

The above constraints can be straightforwardly solved (full details are given in [18]). The result is that the curve (5.3) must take the form

$$y^2 = \prod_{I=1}^N (z - a_I)^2 - p_{N-1}(z) = H_N(z)^2 - \frac{4q}{(1-q)^2} \prod_{f=1}^{2N} (z - m_f), \quad (5.5)$$

where p_{N-1} and H_N^2 are determined uniquely by the requirement that the second equality in (5.5) must hold. The characteristic function is given by

$$\mathcal{F}(z) = \frac{1-q}{2} (H_N(z) + y(z)). \quad (5.6)$$

From this, we can deduce in particular the quantum chiral ring relations and the gauge theory resolvent $\mathcal{R}(z)$. It is sometimes convenient to express the curve (5.5) in terms of the parameters e_I defined by the relations

$$\langle \text{Tr}_F X^p \rangle = \sum_{I=1}^N e_I^p, \quad 1 \leq p \leq N. \quad (5.7)$$

There is a great confusion in the literature when trying to express the Seiberg-Witten curve in terms of these physical parameters. However, knowing the quantum chiral ring it is a straightforward calculation. Introducing the polynomial

$$E_N(z) = \prod_{I=1}^N (z - e_I) \tag{5.8}$$

we find

$$H_N = E_N + \frac{q}{(1-q)^2} \left[\frac{\prod_{f=1}^{2N} (z - m_f)}{E_N} \right]_{\text{Pol.}}, \tag{5.9}$$

where we take the polynomial part of the rational function on the right hand side of (5.9).

The above solution does not depend on the parameter ϵ in (5.1), and thus applies to the $\mathcal{N} = 2$ theory that is obtained in the $\epsilon \rightarrow 0$ limit [7]. Strictly speaking, we have obtained the full set of chiral operator vacuum expectation values. The $\mathcal{N} = 2$ prepotential that governs the low energy effective action can then be straightforwardly derived using Matone’s relation and its generalizations [25]. Matone’s relation itself is a consequence of superconformal Ward identities, as shown by Howe and West in [26]. The result is that the period matrix of the curve (5.5) yields the low energy U(1) coupling constants, as expected.

The theory with $N_f = 2N$ has a β function equal to zero, and thus it has long been conjectured to have a non-trivial S-duality invariance. Accordingly, the original curve for gauge group SU(2), derived in [14], was presented in terms of modular forms for a coupling τ_{eff} . It was soon realized, however, that even though τ_{eff} and the microscopic gauge coupling τ were identical at the classical level, they could not be equal in the quantum theory [27]. Several curves were subsequently proposed for any gauge group SU(N), all modular covariant with respect to some coupling τ_{eff} whose relation with τ has remained unclear (see for example the discussion in [28]). Remarkably, our solution (5.5) has an unambiguous dependence in the UV coupling τ that is not consistent with S-duality. This is in sharp contrast with the $\mathcal{N} = 1^*$ theory, for which the same approach yields results in beautiful agreement with S-duality [1, 29].

5.2 The mass gap and confinement

We start from the pure $\mathcal{N} = 2$ theory, whose curve

$$y^2 = \prod_{I=1}^N (z - a_I)^2 - 4\Lambda^{2N} \tag{5.10}$$

is the limit of (5.5) when the quark masses go to infinity. The limit of equation (5.9) implies in this case that

$$\langle \text{Tr}_F X^p \rangle = \sum_{I=1}^N a_I^p, \quad 1 \leq p \leq N. \tag{5.11}$$

We can go to the SU(N) theory and eliminate the irrelevant global massless U(1) by imposing the tracelessness condition $\sum_I a_I = 0$. For generic values of the moduli a_I , the low energy theory is then a free abelian U(1) ^{$N-1$} theory. There are N special points on

the moduli space where the curve (5.10) degenerates. The behaviour of the low energy U(1) couplings near these singularities shows that a dyon of magnetic charge one becomes massless there. The electric charge of the dyon can be any integer $0 \leq k \leq N-1$, depending on which singular point we consider.

We now turn on the tree-level superpotential

$$W_{\text{tree}}(X) = \frac{1}{2}mX^2. \quad (5.12)$$

If m is infinitesimal, the resulting theory can be seen as a small deformation of the original $\mathcal{N} = 2$ theory. However, the $\mathcal{N} = 1$ theory does not have a moduli space. The moduli of the $\mathcal{N} = 2$ theory must be “frozen” when the superpotential (5.12) is turned on. In [14], Seiberg and Witten assumed that the moduli are frozen precisely at the singular points on moduli space. *This is equivalent to assuming that the $\mathcal{N} = 1$ theory has a mass gap.* Since this is an important point, let us present a simple proof.

Suppose that the $\mathcal{N} = 1$ theory obtained by deforming $\mathcal{N} = 2$ by adding (5.12) has a mass gap. Since the creation of a mass gap is a low energy phenomenon, we can analyze the problem in the framework of the low energy effective action. When $m = 0$, we have a free abelian theory. The only way such a theory can create a mass gap is through the usual Higgs mechanism. But this requires charged fields to be present in the effective action. This occurs only at the singular points on the $\mathcal{N} = 2$ moduli space. Conversely, assume that by turning on (5.12) the moduli are frozen at the singular points. It is then possible to analyze directly the effect of W_{tree} in the effective action using symmetry arguments, and then to show that the massless dyon indeed condenses [14]. Since the field that condenses is magnetically charged, *this also implies (oblique) confinement.* Using analyticity, one can actually compute the dyon condensate for any value of m [30, 9] and show that it is non-zero including in the limit $m \rightarrow \infty$ where we flow to the pure $\mathcal{N} = 1$ gauge theory.

We are now reaching the finishing line. By using the solution of the $\mathcal{N} = 1$ theory with the tree-level superpotential (5.12) described in sections 2–4, we can compute exactly the correlators $\langle \text{Tr}_F X^p \rangle$. Using (5.11), it is straightforward to check that, indeed, *these values correspond exactly to the singular points on the $\mathcal{N} = 2$ moduli space.* From what we have said before, this demonstrates that a mass gap is created and that confinement occurs in the $\mathcal{N} = 1$ super Yang-Mills theory.

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